SQUARE ROOT OF 2 AND THE IDEA OF CONVERGENCE: A SOCRATIC EXPERIENCE

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Abstract.
Our purpose is to reveal the details of the complex logical structure underlying the concept definition of convergence. Our strategy is to facilitate the building of a suitable concept image of convergence by taking √2 as idée fixe and guiding hand in the process of design of a Socratic dialog which is the basis for an interview performed on students of first year college studies. Besides paper and pencil and verbal exchanges, interviewer and interviewee make profuse use of a computer generated tool to cover, on one hand, the arithmetic and visual components of the idea of convergence and, on the other, to recreate the dynamic character of this concept. Success is attained if the interviewee is able to translate the constructed concept image into algebraic and logic terms, for which purpose our experience starts with a fine tuning in quantifiers in common and mathematical language. This educative experience wants to highlight the importance of linking the process of discovery, understanding and conceptualization in theory building and the role played by the computer in attaining this goal in the form of a battery of actions that could be implemented prior to formal mathematical instruction in the classroom.

1. Purpose

√2 is a mathematical object whose existence is readily accepted by students of all levels as the length of the diagonal in the unit square. The purpose of this experience is to introduce the idea of convergence and its algebraic-logic formulation via the challenge of showing that it is possible to calculate √2 as precise as desired. We shall work with first year college students who are acquainted with the intuitive concept of limit but do not yet have any learning experience with mathematically rigorous proofs using formal definitions.

Once accepted that √2 should be a number, we design a strategy of understanding its nature. We proceed by iteration providing computer-generated visual and numerical rational approximations by turning the progress of our arithmetical calculations and bi-dimensional figures into an algebraic-logic statement which encapsulates the idea of how to jump from approximations to limiting value, therefore putting to rest a dynamical process. In order to succeed we need to build up a progressive understanding of the relationship between the ‘neighborhood’ ε (‘closeness’) and the index N (location in the sequence of approximations) in defining the limit of a sequence by revealing the details of the complex logical structure underlying this definition (the role played by quantifiers and their order of appearance) and hence encourage the development of an ability to associate an informal statement with its logically structured formal statement in mathematical contexts.

The experience which is going to be described allots considerable effort to a single definition, namely that of convergent sequence. One could object to spending so much time with a single definition, but definitions matter and make sense when a point is reached which makes distinctions necessary. To make definitions without motivation is more likely to cause confusion. Do most people learn a new concept most efficiently by being exposed to elegant definitions? Is it efficient to show students forthright the logical structure of mathematics and hope they catch on? (see Edwards & Ward, (2004)). Concerning the definition of convergence, this educative experience reassured us on the idea that, in an effort to create an illusion of clarity, stating a definition on the blackboard as succinct as possible and proceeding with the typical cascade of propositions and theorems (as usually done in the classroom) is a recipe for failure: it is clear to us that even the most gifted students struggle considerably with understanding the meaning of the concept of limit through its definition (Edwards (1977)) and reconciling it with their previous ideas of what converge may mean. Many of the difficulties that university students may have with formal mathematics might well be viewed as stemming partly from an unawareness that mathematical definitions tend to be analytic (whatever the definition says it is, nothing more and nothing less), rather than synthetic (descriptions of something that already exists). What we shall try to show is that the introduction of advanced dynamic topics such as the limit concept can be facilitated by relying on graphical and numerical explorations, and on techniques of an algebraic nature and that transition towards more formal approaches (which take place at the university level) represents a tremendous jump, both conceptually and technically.

2. Convergence

There are plenty of articles devoted to the task of studying how limits are understood by students. Learning the concept of limit informally, students can be helped to construct images about the infinite process of a sequence itself (Navarro & Pérez Carreras, (2006); Navarro & Pérez Carreras, (2011). Examination of primary intuitions related to the notion of limit of a numerical sequence in secondary school pupils who have not yet been exposed to its more formal definition can be found in Sierpinska (1985 and 1987) and Navarro & Pérez Carreras, (2006) where considerable divergences between convictions and definition are pointed out. An introduction to the idea of convergence via decimal expansions can be seen in Navarro & Pérez Carreras (2010) showing conflicts arising from sequences seen as indefinite and ongoing processes without considering them as the result of such processes. In Robert (1982 and 1983) several models of convergence of numerical sequences in university students are presented. Partially similar observations were made in other writings dealing with understanding the limit of a sequence by pupils and students (Davis & Vinner, (1986); Tall & Schwarzenberger, (1978). Students' conceptions related to the notion of limit of a function can be seen in Cornu (1981 and 1991), Ervynck (1981), Ferrini-Mundy & Graham (1994) and Williams (1991). Students' images of limit are pointed out as having influence on understanding the ε–N definition of the limit of a sequence and the complexity of its logical structure makes it difficult to understand why the quantifiers should be described in such a way, Cornu, (1991); Mamona-Downs, (2001); Roh, (2009). Even if this definition has been mastered, students tend to continue to use images of limit that are associated with their previous learning experiences, Pinto & Tall, (2002); Przenioslo, (2004).

3. Our approach

According to the constructivist perspective which focuses in individual thinking and the continuing act of creating learning opportunities, we prefer to accomplish our goal via an appropriate Socratic interview design aided by a computer generated tool. In its design we paid attention to many of the recurrent themes described in the research literature.

Why an interview as a Socratic dialog? The dialog provides motivation and tests the solidity of those ideas and arguments which the student brings and that are based in prior educative experiences when challenged to dig out and verbalize their own beliefs throwing light in his way of understanding things mathematically. We want them to become active participants in the creative act making statements not to fit a preexisting pattern, but because they mean them. The most delicate aspect of the dialog is how to fight the attitude of most students which are comfortable with inconsistencies, contradictions and competing meanings which apart from their previous shortcomings are the main cause of failure in completing the interview. We are not interested in what students can do naturally, but in what they can do accompanied by instruction. We want to guide them in a journey of discovery and inquiry, not random but purposeful, testing at every significant stage of the experience the foundations of their beliefs or their ways of reasoning, very much in the Socratic spirit, and to explore how far progress can be made in providing meaningful information by using colloquial language admitting a certain degree of ambiguity present, which hopefully allow students to invest meaning in the problem studied, while manipulating a visualization and computational tool.

Why a computer generated tool? There are several reasons. Cottrill et al., (1996) concluded that the formal concept of a limit is a dynamical schema and not a static one. In Calculus courses, the student is asked questions about dynamics of the real-world situation that requires extracting dynamic information from a static graph. Surely it ought to be easier to extract dynamic information from a dynamic presentation. Research studies state that students show quite surprising ways in interpreting dynamic figures, which seemed to differ from the classical paper-and-pencil representations. If one agrees that mathematics is at least as much a visual undertaking as it is a symbolic and verbal one, the flexibility of perceiving a concept in different representations can be considered to be proof of understanding this concept. We want to create opportunities in order to encourage the construction of flexible and connected knowledge because not having this flexibility results in failure to make the right connections. Besides, it provides mathematics with the experimental factor leading to the formulation of conjectures, examples and counterexamples: unknown relationships are 'discovered' by the student through his direct relation to the software. Moreover, a very promising aspect of technology-based learning is to utilize the principle of simultaneous activation of conceptual and procedural knowledge. Although it is unclear to us how manipulative aids affect cognitive functioning, we are convinced that, at the very least, its use somehow provokes less disappointment if the machine proves them wrong, favors the appearance of conjectures and frees students' thought processes. Moreover, the generation of images by computer promotes the integration of the separate components of the item in question and accessing parts of the information encoded in memory prompts the retrieval of all other pieces of information contained in the image. Since a person is only able to see what he really understands, a well structured dialog helps integration.

Why $\sqrt{2}$? Because it is the first `problematic' number to appear in connection with elementary geometry and gives rise to (deep) ideas such as incommensurability; the quest to determine it with increasing rational accuracy promotes the design of algorithms which can easily be visualized with our tool allowing the presentation of valid and non valid algorithms and, among those which are valid, the idea of efficiency of execution. Last but not least, allows the appearance of the idea of convergence and $\sqrt{2}$ appears as the limit of a sequence of iterates. All those ideas extend easily to other situations where any number may appear as the limit of a sequence.

The cognitive structure associated to a mathematical concept which includes all mental images, visual representations, experiences and impressions as well as associated properties and processes is called concept image (Tall & Vinner, (1981). Learning, understanding, applying and developing mathematical concepts involves the construction of this kind of structure in the mind (Noddings, (1990); Piaget, (1977); Artigue, Batanero & Kent (2007)). By means of a semi-structured clinical interview, we shall provide the means for the construction of a solid concept image of convergence stepping thoughtfully through the micro-development of it which incorporates visual, numerical and algebraic connotations through the study of a non-routine problem. Our aim is not to develop a substitute for the concept definition (the conventional linguistic statement precisely delimiting the frontiers of application of the concept, Tall & Vinner, (1981), but to describe a battery of actions that could be implemented prior to formal mathematical instruction in the classroom with the purpose of, on one hand, to obtain a detailed understanding on how students think (which should help us in preparing our classroom presentation) and, on the other, construct a suitable concept image which does not distort the desired concept definition, provoke the need for it and, finally, perform a transition to it.

Around twenty individual interviews were carried out and each successful one took one hour and a half time. Students were selected randomly from those reading Calculus at engineering schools at the University of Seville and they agreed to the audio recording of the interviews and also to the use of their corresponding transcriptions in our analysis. Their previous upbringing in mathematics was the usual one provided by the Baccalaureate in Spain. Not all interviewees were capable of completing the interview; as a matter of fact, only six of them (less than one third) finished it satisfactorily, but those performed substantially better than their peers in the traditional examination procedures in Calculus probably because the experience encouraged them to use the computer as a tool when dealing with mathematical problems. As an argument for the defense of such an approach as compared with more conventional ones we have to stress that although it is always possible to tell if students have mastered a specific topic at a specific time, it is doubtful that in the traditional context we arrive to a clear understanding of how students accomplish, of the intermediate stages of understanding they may pass through along the way, of what assisted or impeded that learning, or what understanding will remain. Those goals can be reached by allotting more time to concept development which, in our experience, pays off handsomely. The intensive use of interviews tells us that, as a general rule in teaching nowadays, we should become better listeners.

To show how the experience went for those students, we have assembled a single interview with a fictitious student, whose answers come from a variety of students who were able to reach the interview's conclusion and has been heavily edited to make it readable, avoiding false starts, silences, clumsiness and hesitations, although what is given below is a fair representation of their clarity of thought and adaptability. In order to keep this article within reasonable bounds, dysfunctions do not show in the following transcription of our interview, since the responses belong to the more gifted students. Anyway, we point out the main causes of failure of the others along the interview.

4. The tool

The tool is an interactive screen and the user does not need to have previous knowledge on the program (MATLAB 6.1) used to design it. The screen shows two windows (Figure 1), a graphical (GW) and a computational one (CW): GW allows the plotting of functions and CW shows three columns corresponding to order, iterates and the difference in absolute value between iterates and $\sqrt{2}$ as calculated by the tool. When very small quantities are to be used as indicative of scale, the notation showed by the screen is the usual one of a MATLAB program ("e-00k" means 10^{-k}). The number of digits present in CW can be chosen.

Several buttons allow to write the analytic expression of the functions to be considered, start an iterative procedure ("Activating GW") from a chosen starting abscissa point and choose the number of iterations to be performed which will show in GW as a spiral encroaching some value or a staircase approaching it or both going away from the desired solution to our iterative procedure. We keep equal scales for both abscissa and ordinate in order to be able to make predictions on the success of our iterations as related to inclination close to the desired point.

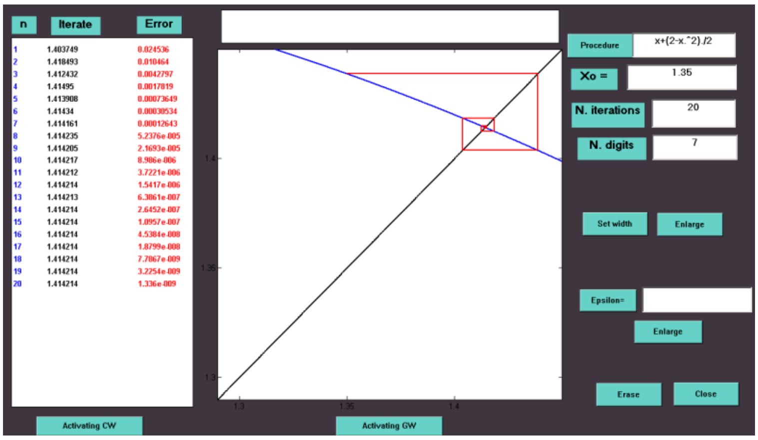


Figure 1

Zooming capability is restricted around ($\sqrt{2}$, $\sqrt{2}$) and its amount is selected with "Set width" (with the mouse, choose the length of the half of the side) or we can write the half of the length of the side on the corresponding edit text control of the screen (Epsilon) (see Figure 2). When a zoom is done, only a portion of the spiral/staircase will appear: a message will be activated indicating the scale and how many approximations are left outside the visual field (Figure 3).

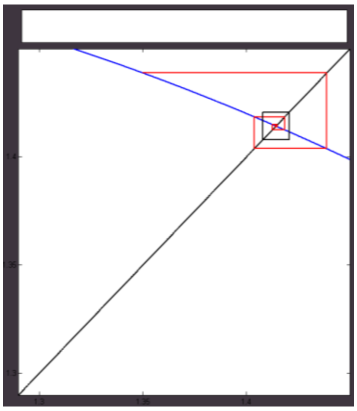


Figure 2

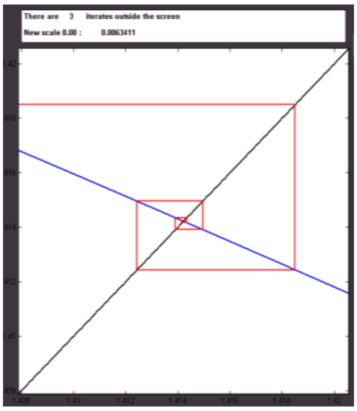


Figure 3

We may add more iterates of the process and/or perform new zooms. The pushbutton "Erase" allows us to start again with the original scale, changing the function and/or the initial point or number of digits allowed.

5. The interview and student's responses

5.1 Phase 0 (Prerequisites)

We start our experience with considerations about quantifiers in common language. It goes like this: all known human languages make use of quantification; for example, in English, *every* test in my recent exam was flawed; *some* of the students in the classroom have a pocket calculator; *most* of the students I talked to didn't have a clue; *everyone* in the classroom has at least one complaint against the teacher; there was *somebody* in the class that was able to correctly answer every one of the questions I submitted; *a lot* of students are smart (the words in italics are quantifiers). We ask the interviewee to interpret all of them. We point out that these examples suggest that the construction of quantified expressions in natural language can be syntactically very complicated, but we offer hope by stating that the study of quantification in natural languages is much more difficult than the corresponding problem for formal languages and we offer as explanation that this comes in part from the fact that the grammatical structure of natural language sentences may conceal the logical structure.

We make him aware that the language used in the teaching of Mathematics has its own conventions and assigns a degree of precision to words and logical terms which are not coincident with the ones used in natural languages in order to remove ambiguity and context-dependent meanings. Since one of the purposes of teaching is to fix students in context, we clarify that we are going to use a kind of semi-dialect obeying rules that have to be mastered, because otherwise whatever understandings students may have of quantified statements may not transfer easily to mathematical settings.

Fortunately, for mathematical assertions, the quantification process is syntactically more straightforward and we offer several easy, short and reasonably natural instances as the ones above and also several where mathematical formulas mix symbolic expressions for quantifiers, with natural language quantifiers such as '*for any natural number x, ...*' or '*there exists an x such that ....*' checking in the process if he is in possession of or has developed a rudimentary but accurate feeling for syntax to concentrate afterwards in semantics, once we have informed him that the traditional symbol for the universal quantifier "for any" is "∀", an inverted letter "A", and for the existential quantifier "there exists" is "∃", a rotated letter "E". First we go from semantics to syntax and then the other way round to concentrate later in the order of appearance of the quantifiers and what difference it makes.

Pr.: Let us state in plain English the following sentence which may be true or not: There is not a largest integer number. Do you understand the statement and, if so, do you think it is true?
St.: Indeed, it is true since if I consider an integer m, I can always construct a larger one simply by adding the unity.
Pr.: Can you formulate the statement starting with "*For any integer m, there exists ...*"
St.: For any integer m, there exists another integer larger than it.
Pr.: This other integer, does it have a name?
St.: Let us call it k. Now we have: For any integer m, there exists another integer k such that k>m
Pr.: We are almost done. Can you shorten the statement by providing quantifiers for it?

St.: $\forall m\ \exists k\ k>m$.

The transition from verbal to formal is completed. What about the other way round?

Pr.: Look at the structure of the statement: in our case it is '$\forall m$, something'.
St.: Something meaning '$\exists k\ k>m$'.
Pr.: Yes. To really understand the whole statement, you need to first understand what that 'something' is saying. So you will have better success understanding the full statement if you first understand the shorter statement '$\exists k\ k>m$' then realize that the whole statement says that the shorter statement is true for all m.
St.: Then the whole statement is obviously true.

Now we inquire about the importance of the order of appearance of the quantifiers in our statement about integers

Pr.: Now suppose I turn the order of the quantifiers '$\exists k\ \forall m\ k>m$' which in plain English means …?
St.: (hesitating) '$\forall m\ k>m$' means that k is larger than all possible integers. The statement is saying that there is some integer that is larger than every integer.
Pr.: The only difference between statements is the order of the quantifiers, but the meaning is much changed. In fact, the statement is...
St.: False. Thus, the validity of a sentence depends on how the quantifiers are placed?
Pr.: I am not saying that one is the correct order and one is the incorrect order. They are just statements that say different things. The order you use depends on what you want to say.

In order to provide a routine to ascertain whether a statement on integers involving quantifiers is true or false, we propose to play a game: A concrete statement with quantifiers is provided. The interviewer attacks the statement and the student defends it; to each variable with the universal quantifier attached the attacker assigns a numerical value and the defender does alike for each variable with the existential one attached to protect the statement. The order of quantification determines the order in which the players do their movements. The statement is true or false depending on whether the defender or attacker wins the game, respectively.

Pr.: Statement is '$\forall x\ \exists y\ x+y=3$' for integers x and y. According to the quantifiers order and the rules of the game, I shall start attacking with 16.
St.: I defend with -13.
Pr.: Can you defend the statement reacting to any choice I can possibly provide?
St: Certainly; I win the game in any possible case and hence the statement is true.
Pr.: Now we turn to the statement '$\exists y\ \forall x\ x+y=3$'. Who starts?
St.: According to the rules, I do. I defend with 2.
Pr.: I attack with 2. I win, hence the statement is false.

Report. Around one third on the interviewed failed our prerequisites and we decided not to continue with them; most of them would probably be capable of sailing satisfactorily through Phases 1 and 2, but they had no chance going through Phase 3, which is our objective. Although all of those who failed understood the importance of constructing a partially formalized language to avoid vagueness and misunderstandings, they were unable to leave behind a vague understanding of the meaning of both quantifiers and had a poor performance when going from semantics to syntax which showed a deficient High School training, since what is really problematic, as observed by several authors, is the other way round: students are reluctant to use the syntax of a statement in order to interpret it. That obstacle is not significant in our study, since our purpose is to build an understanding and then go to its syntax. Those who survived the first block of questions and our previous considerations sailed comfortably through the other two. The verification through a game posed no threats to them.

5.2 Phase 1 (arithmetic nature of $\sqrt{2}$)
We remind him of the concept of prime number and The Fundamental Theorem of Arithmetic - every number is uniquely (up to the order of factors) representable as a product of prime numbers. Leaving the realm of integers, we remind him as well of the concept of rational number as a quotient p/q, q≠0 with integers p and q in lowest terms and known facts such as (i) integers can be considered as rational numbers, (ii) the decimal expansion of a rational number always either terminates after finitely many digits or begins to repeat the same finite sequence of digits over and over and again, (iii) any repeating or terminating decimal represents a rational number. Those decimal expansions show the use of rational numbers for lengths measuring purposes. Whether every length is measurable by rational numbers and whether we should look for a broader concept of number is our first port of call.

Pr.: Suppose we have a square of area 2. Calculating the length x of its sides means solving the equation $x^2 = 2$. What can you tell me about the numerical value of x?
St.: (using his pocket calculator) Length is $\sqrt{2}$, which is close to 1.414
Pr: Right. Does the symbol $\sqrt{2}$ stand for a number?
St.: What else can it be? It stands for a length.
Pr.: Length and number are synonyms? Is $\sqrt{2}$ a rational number, that is, expressible as a fraction?
St.: Every length has a number associated to it and the other way around. Concerning your second question the answer is that I guess so, since, according to my calculator, it has a decimal expansion.
Pr.: Hmm…, rational numbers and decimal expansions are synonyms again?
St.: Well, you mentioned that before. Rational numbers allow a decimal expansion which is obtained by repeated division; every division I perform, I get a new digit.
Pr.: Right, but is it another way of understanding a decimal expansion independent of from where it comes from, let us say, from a rational number and the associated process of repeated division?
St.: A decimal expansion by itself can stand for a measurement of, let us say, a length.
Pr.: How so? Imagine we are talking about 0.111…
St.: By dividing the unit segment in ten equal parts I locate 0.1, each part in ten equal parts and I locate 0.11 and so on I can locate 0.111… and hence it can be understood as the measure of the length, which in this case is 1/9. Conversely, the same idea also works.
Pr.: Are you saying that every length can be measured by a decimal expansion and the other way round?
St.: Yes. I can identify decimal expansions with lengths.
Pr.: Can be seen also as numbers?
St.: I understand the same idea whether you call it number, length or decimal expansion.
Pr.: Let us assume for the time being all those identifications and your intuitive idea of what a decimal expansion is. Let me ask you a question: If I assume $\sqrt{2}$ to be a rational number, let us

say $\sqrt{2}=p/q$ for integer p and q, and if somehow I reach a contradiction, what would your conclusion be?
St.: If the logic behind your 'somehow' is sound, I would say that your assumption is untrue.
Pr.: Right again. Let $(p/q)^2 = 2$ for some integers p and q. Then $p^2 = 2q^2$. Factor both p and q into a product of primes. p^2 is factored into a product of the very same primes as p.
St.: Each taken twice.
Pr.: Indeed. Therefore, p^2 has an even number of prime factors.
St.: So does q^2.
Pr.: Therefore, $2q^2$ has an odd number of prime factors. Contradiction!
St.: Hence $\sqrt{2}$ is not a rational number; but I know it to have a decimal expansion …
Pr.: Albeit not a periodic one.
St.: Ah! 'Periodic' makes the difference then. How can you be sure that a decimal expansion is not periodic? No matter how many digits you throw, the period may appear later on. In other words, are there non-periodic decimal expansions which does not terminate?
Pr.: You can construct easily as many as you want. Consider 0.**2**3**22**3**222**3**2222**3… and so on. No period will ever appear. Now you can add $\sqrt{2}$ to the list and consider the term 'number' to be synonymous of decimal expansion, periodic or not, or of length of a segment.
St.: (puzzled) Right. But with a periodic expansion I can always recuperate the rational number which produces it and therefore I can calculate the length represented by it in an exact way. Mathematics is supposed to be an exact science. How to proceed with non-periodic ones such as $\sqrt{2}$?
Pr.: That is exactly, no pun intended, the problem we are going to deal with.

We explain that our goal is twofold: on one hand, we are trying to know how to reach such approximate values and, on the other, we shall try to understand what $\sqrt{2}$ is. Using the calculator to approximate $\sqrt{2}$ by rational numbers looks as a very fast and efficient method to evaluate $\sqrt{2}$. However, we point out that the simplicity is rather deceptive. In fact, there is a complexity in the computation: the formulas commonly used for calculating $\sqrt{2}$ only make use of additions, multiplications, and divisions and are hidden behind the '√' key on the pocket calculator.

Report. We were left with fourteen students with a previous knowledge of Algebra, Pre-calculus, an intuitive idea of what a decimal expansion means (as a step by step procedure not necessarily linked with repeated division) and a fair capability in translating common to formal language and the other way round, at least, related to simple statements. Roughly half of them posed some resistance in accepting the proof of the irrationality of $\sqrt{2}$ by reductio ad absurdum but they could be convinced of the suitability of proceeding that way when trying to prove that something is not what it is supposed to be, although the last bit of suspicion couldn't be removed; we didn't press the point. The identification of rational numbers and decimals expansions was fairly common to all of them (which is only natural because this was their first encounter with irrationality) and the existence of non-periodic non-terminating decimal expansions came as a surprise; they recuperated but felt somehow uneasy because they started to realize that 'decimal expansion' is something that should be made more precise once repeated division was eliminated from the landscape. The questions arose: if $\sqrt{2}$ is not a rational number, what is it? They accepted that it should be a number (enlarging therefore their previous idea of number as identified with rational ones), that it should have a decimal expansion and that it should be possible to calculate each and every digit in it. In absence of other known procedures for them, if repeated division was no longer a viable procedure to obtain any digit in its decimal expansion, what is the alternative? Some of them posed the question: if my calculator provides a fair number of digits, which for practical purposes should be good enough, why should we press the point further? We answered that mathematics is the art of explanation and it should be as exact as the human mind is capable of and we promised a gain in insight with far reaching consequences if they accepted to travel further with us: a good problem does not just sit there in isolation, but serves as a springboard to other interesting questions. No student failed Phase 1.

5.3 Phase 2 (approaching √2 visually and arithmetically through iteration)
In order to understand how to approximate by rational numbers and what the exact value of √2 may mean, we lead them to investigate feedback in mathematics – what happens if you take the output from a function and feed it back into the function again and again. Repeatedly solving an equation to obtain a result using the result from the previous calculation is called **iteration**. The procedure is used in mathematics to give a more accurate answer when the original data is only approximate.

We question our interviewee on what the term 'function' means for him. He has been already introduced to this notion and hence is expected to manipulate it with confidence. Since he will always object to the absence of a well-defined rule, that is a function *f* needs that a variation in *x* has to be systematically reflected in the *y* and that manipulation carried out on *x* produces *y*, we shall stick to this (incomplete) view of what a function is. Our tool will deal with the computational and visual aspect of the notion identifying functions with computer programs.

> Pr.: Iteration is repeatedly performing a routine. For instance, if you want to add up all the whole numbers less than 5, you would start with 1 (in the 1st step), then (in step 2) add 2, then (step 3) add 3, and so on. In each step, you add another number, which is the same number as the number of the step you are on.
> St.: Each step leads onto the next, like stepping stones across a river?
> Pr.: Right. The only part that really changes from step to step is the number of the step, since you can figure out all the other information (like the number you need to add) from that step number. This is the key to iteration: using the step number to find all of your other information.

He manifests impatience and enquires about how all that applies to the calculation of √2. We explain that our way is similar to a journey through unchartered territory: we need a map (set of instructions or procedure to be executed) and that we have to start somewhere. Concerning where to start he points out that a known approximate value such as 1.4 would do. We question him on where to start if the '√' key were not available: we cannot avoid the fact that he knows already 1.4 as a good candidate; using his calculator, he squares 1.4 to show us how close it is to the actual value and its starting point status is justified. Concerning procedure, he comes quickly with the idea of squaring 1.4 and 1.5 to obtain bounds to the value of √2; he proceeds further: why not taking the mean of 1.4 and 1.5 and looking, whether its square (1.45^2=2,1025) is larger or smaller than 2. Now he takes the mean of 1.4 and 1.45 and so on (producing nested intervals). By trial and error, he proceeds to fix 'right' decimals.

> Pr.: You have devised an arithmetic procedure similar to tuning a guitar or parking a car between two parked cars …

St.: Which gradually produces improved approximations of the result, yes, but it is tiring and very time consuming, even with my calculator.
Pr.: Even if we better leave the dirty work to a machine, for that purpose we need to design an algebraic procedure you understand (in opposition to the mystery of the '√' key) whose task is to produce gradual improved approximations and quicker than the one you've used.
St.: Yes, but how?
Pr.: The key idea is by using the output of one step as the input of the next step in the procedure. Which algebraic entity transforms numbers into numbers?
St.: Hence under the term procedure you mean a function.
Pr: Right.
St.: Which function or procedure do you have in mind?
Pr.: We shall come to that, but for the time being, suppose such a procedure is already at your disposal. Call each output **iterate**.
St.: One iterate plugged as an abscissa into the procedure produces the next iterate, an ordinate, which in turn is considered as an abscissa, plugged again and so on?
Pr.: You got it. Thus, a sequence of numbers appears and it is called an **orbit**. Some orbits can get as close as desired to the theoretically exact value of √2. Be aware that I have used the quantifier 'some'.
St.: By what you mean that not 'all' orbits are of the approximating type. Supposing it is so, does it depend on the choice of procedure?
Pr.: Different procedures produce different orbits. But observe that, even if we stand with a certain procedure, different orbits may be generated depending on where we start. Once we choose a starting value, the orbit is unmovable.
St.: In iteration, a problem is converted into a train of steps that are finished one at a time, one after another. But, how many times steps do I need to take, that is, how long should the orbit be?
Pr.: The criterion for terminating the iterations is usually met when an appropriate error bound (and therefore the error) is found to be sufficiently small, thereby assuring the desired accuracy.
St.: Thus, iteration provides approximations to √2 but it does not tell us what number √2 is in an exact way.

Now we come to the choice of procedure which must be somehow related with the equation $x^2 = 2$ generating our problem. We point out that nothing of much interest is obtained by just plugging 1.4 into that equation, as done before. He stares helpless and we provide a hint: manipulate this equation to allow the game of input, output and feedback by having 'x' in both sides of the equation: we take paper and pencil and we show how to do it: For instance, rewriting the equation as $x = 2/x$, using the right hand side $2/x$ as procedure and choosing the input value $x_1=1.4$ we calculate the output $2/x_1$. Calling this value x_2 which was our output we treat it as the next input to obtain the output $x_3=2/x_2$ and so on. Summarizing: leaving x on its own on the left, the left hand side x becomes x_{n+1} and the right hand side x becomes x_n. The equation is now in its iterative form. We start by working out x_2 from the given value x_1; x_3 is worked out using the value x_2 in the equation; x_4 is worked out using the value x_3 and so on.

Now we go visual. According to Piaget, going visual is usually more difficult and harder to teach because you only see what you understand; thus a detailed explanation of the tool takes place. We activate GW showing the graphs of the bisecting line of the first quadrant x and the 'procedure' $2/x$. We explain why (√2, √2) is placed at the intersection of both graphs. We choose our first input as 1.4 and proceed with one iteration to see where x_2 lies. We use the zoom facility to see what is going on. We explain that for comparison purposes we want to see x_2 in the x-axis which means going through the bisecting line of the first quadrant and the bi-dimensional materialization of the orbit, a spiral in this case, begins to take form (Figure 4a). We proceed with further iterations and hence more steps in the spiral (Figure 4b). Visually, we ask him to discuss the relationship between the bi-dimensional curve generated by the procedure (the spiral) and the location of each iterate to which he will point out that iterates should be in the x-axis and are the abscissas of the points where the spiral cuts the graph of the procedure. In order to see the numerical values of each iterate we ask him to activate CW (see Figure 4c)

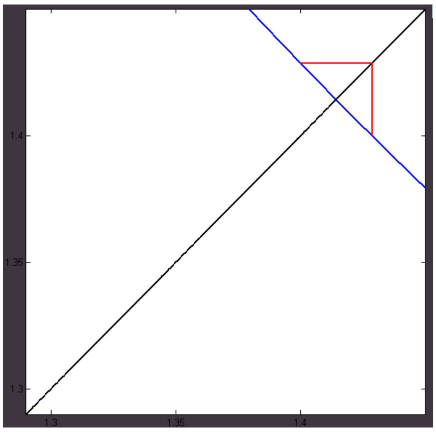


Figure 4a

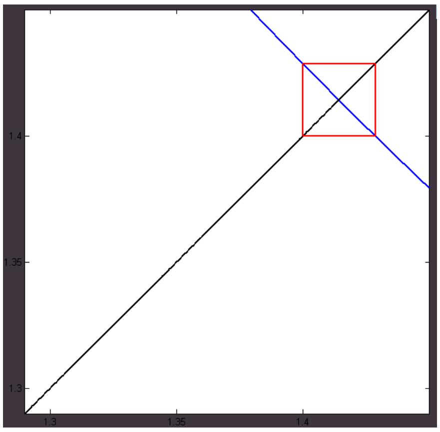
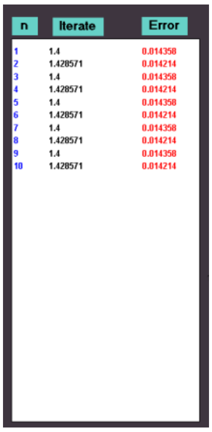


Figure 4b



n	Iterate	Error
1	1.4	0.014358
2	1.428571	0.014214
3	1.4	0.014358
4	1.428571	0.014214
5	1.4	0.014358
6	1.428571	0.014214
7	1.4	0.014358
8	1.428571	0.014214
9	1.4	0.014358
10	1.428571	0.014214

Figure 4c

Pr.: Visualizing the procedure you have our starting value on the left of $\sqrt{2}$ and the first iterate on the right side. What happens next?

St.: Quite frustrating! Those values repeat themselves and nothing better than 1.4 is obtained.

Pr.: You entered a loop. Who is to blame?

St.: It has to be the procedure; there is nothing else. We have to look for a different one.

Pr.: Right. Remember the origin of our problem: calculating the length of the side of a square of area 2. Reflect that if a rectangle has area 2 then its sides have to be x and $2/x$. According to our 'tuning' idea, the average of these sides is thus closer to the side of a square of area 2 than was either side of the original rectangle.

St.: Assuming the original rectangle was nearly square.

Pr.: Right. Why don`t we use then $\frac{1}{2}(x+2/x)$ as procedure using our tool? Check that the equations $\frac{1}{2}(x+2/x) = x$ and $x^2 = 2$ are the same equation.

St.: (doing so) Right. What I see now looks not as much as a spiral, but as a staircase approaching the intersection point ($\sqrt{2}$, $\sqrt{2}$) (see Figure 5).

Pr.: Good observation. Why a staircase and not a spiral?

St.: (hesitating) Almost all iterates are on the same side of $\sqrt{2}$ and there the staircase begins to take shape. If I proceed with more iterates zooming for a better view (Figure 6) the trend continues and at each step I am closer to the intersection point.

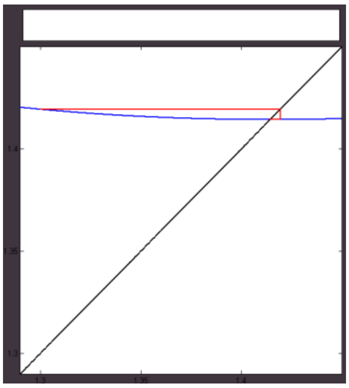


Figure 5

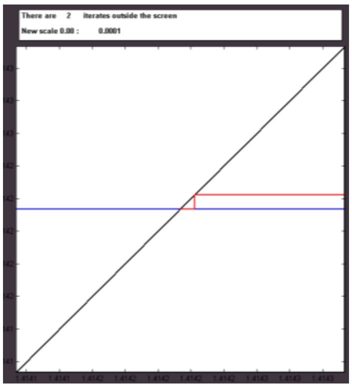


Figure 6

We need to enlarge the pool of possible procedures at his disposal to facilitate the discovery of visual characteristics of the procedure which play a crucial role in deciding whether it provides a useful orbit or not.

Pr.: What happens if you choose $x+½(2-x^2)$ as procedure?
St.: It seems to come from nowhere. Why this procedure?
Pr.: Right. It doesn`t come from any geometrical intuition, but note that it is a valid procedure since our original equation and $x = x+½(2-x^2)$ are the same and allows iteration.
St.: (checks and uses the tool, see Figure 7) Now I have a spiral encroaching (√2, √2) and their abscissas should provide better and better approximations to the value of √2.
Pr.: Now check $2x-2/x$
St.: (doing so starting with 1.41, see Figure 8) Again is a valid procedure. Now I see a staircase going the wrong way.
Pr.: As you have seen, some procedures seem to work to our advantage and some don't (presenting him in a leaflet the figures corresponding to our four former procedures). Can you provide a geometrical argument to explain those different behaviors?
St.: (taking time) It seems that what matters here is what happens when we start the orbit close to √2; I mean the shape of the graph of the procedure around the intersection point.
Pr.: By shape you mean …?
St.: Very flat, it seems to work; take it very steep, it doesn't.

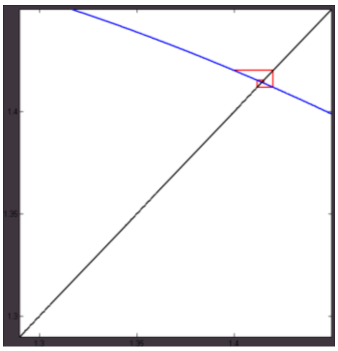


Figure 7

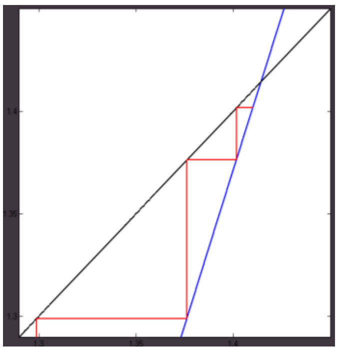
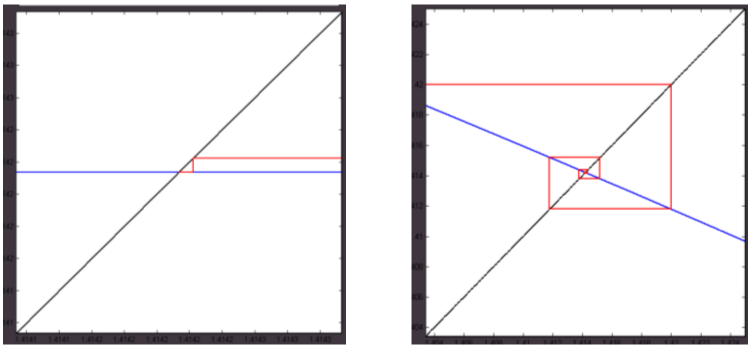


Figure 8

Unrelated to the specific problem we are dealing with, we ask him to use paper and pencil to draw the graph of the bisecting line and several other curves which could act as procedures generated by other problems starting with a straight line: he is able to reproduce manually what the tool does and it is quickly confirmed that the inclination of the bisecting line acts as a frontier between working and not-working procedures. We propose other curves to confirm if staircases or spirals perform as conjectured being tha smallness of the inclination of the procedures (understood as slope of the corresponding tangent line) the key to operability.

Pr.: From our former two working procedures, which one acts faster?
St.: Faster means?
Pr.: Requires the least number of iterations to approach √2 with a desired degree of closeness.
St.: (looking at GW, see Figures 9) Well, the one exhibiting the smaller slope in absolute value.



Figures 9

Report*.* The general idea of feedback was understood by all remaining students but its mathematical treatment, that is, its articulation via the concept of function, posed real problems to five of them although patience on our side paid off initially. Understanding the idea of procedure and having the capability of implementing one are different but related abilities; not surprisingly, those same students had also difficulty thinking of a procedure to approach√2 by trial and error squaring known approximations but reluctantly accepted the proceeding when told, not because they didn't grasp it but because they had not been able to figure it out by themselves. Unfortunately, it came the time where we lost them along the interview due to their inability to identify the idea of iteration with what the bi-dimensional setting the tool is capable of providing; they were unable to extract information from the visuals of our tool, distinguish between iterates and spirals and verbalize what the behavior of the spiral meant in connection to the idea of approximation. Some of those five who were able to jump to the bi-dimensional setting could not provide a clue on why certain procedures worked as desired and others not. Even if they had a reasonably good upbringing in arithmetic, algebra and basic logic they acted as a typical case of what damage a High School curriculum does when mathematics is taught emphasizing syntax over semantics, favors the absence of visual considerations related to thought processes and avoids mathematical assistants in the classroom. The remaining nine students showed signs of struggle along the interview but were able to surmount difficulties and went unscathed to Phase 3.

5.4 Phase 3 (to the limit)
This Phase deals with the formulation of the concept definition of convergence of a sequence of numbers (those of the successive approximations obtained in an effort to be more and more precise about the 'real' value of √2) and goes in three steps. Since the devil is in the details we have to proceed carefully. First, use the numerical and visual capabilities of the tool to verbalize the idea of

'getting close' or 'approach' where logically conditioned statements have to be expressed with precision. Second, perform a cleanup of verbal terms suggested by visualization by substituting them by logic and algebraic terms and construct a chain of logically conditioned inequalities. Third, once arrived to a mathematical formulation of the notion of convergence, check its validity by means of a game where the tool plays a role again.

First Step: We point out that if exact digits in the expression of √2 is what is wanted, the use of the procedure *½(x+2/x)* is very much recommended for its efficiency and speed, therefore substituting the 'mystery' of the '√' key in the calculator by an understandable way of performing the task. We concentrate on CW for this procedure and select several iterates to be seen and we ask him to note that the number of digits which are stable duplicates at every iteration which gives him a numerical appreciation of the fastness of the procedure (see Figure 10).

n	Iterate	Error
1	1.4166666666666665	0.085786
2	1.4142156862745097	0.0024531
3	1.4142135623746899	2.1239e-006
4	1.4142135623730949	1.5947e-012
5	1.4142135623730949	2.2204e-016
6	1.4142135623730949	2.2204e-016
7	1.4142135623730949	2.2204e-016
8	1.4142135623730949	2.2204e-016
9	1.4142135623730949	2.2204e-016
10	1.4142135623730949	2.2204e-016

Figure 10

Now we arrive to our main purpose, mainly the study of how to verbalize and formalize the idea of 'getting close' with the aid of visualization. We turn our attention to the procedure $x+½(2-x^2)$ instead of the more natural *½(x+2/x)* for the last one is too fast to allow us to see what is going on in GW. We use again the tool with only CW activated, allowing six digits to be shown.

Pr.: How many iterates are needed to reach the value 1.41421?
St.: Nine (Figure 11)
Pr.: (allowing eight digits) How many to reach 1.41421 35?
St.: Thirteen (Figure 12)
Pr.: (allowing nine digits) How many to reach 1.41421 356?
St.: I need seventeen iterations to reach 1.41421 356 (Figure 13)

n	Iterate	Error
1	1.4118	0.0057864
2	1.41521	0.0024136
3	1.4138	0.00099682
4	1.41438	0.00041339
5	1.41414	0.00017115
6	1.41424	7.0906e-005
7	1.4142	2.9368e-005
8	1.41422	1.2165e-005
9	1.41421	5.0388e-006

Figure 11

n	Iterate	Error
1	1.4118	0.0057864
2	1.4152104	0.0024136
3	1.4138002	0.00099682
4	1.4143847	0.00041339
5	1.4141427	0.00017115
6	1.4142429	7.0906e-005
7	1.4142014	2.9368e-005
8	1.4142186	1.2165e-005
9	1.4142115	5.0388e-006
10	1.4142144	2.0872e-006
11	1.4142132	8.6453e-007
12	1.4142137	3.581e-007
13	1.4142135	1.4833e-007

Figure 12

n	Iterate	Error
1	1.4118	0.0057864
2	1.41521038	0.0024136
3	1.41380017	0.00099682
4	1.41438471	0.00041339
5	1.41414266	0.00017115
6	1.41424293	7.0906e-005
7	1.4142014	2.9368e-005
8	1.4142186	1.2165e-005
9	1.41421148	5.0388e-006
10	1.41421443	2.0872e-006
11	1.4142132	8.6453e-007
12	1.41421371	3.581e-007
13	1.4142135	1.4833e-007
14	1.41421359	6.144e-008
15	1.41421355	2.5449e-008
16	1.41421357	1.0541e-008
17	1.41421356	4.3664e-009

Figure 13

Pr.: (leaving nine digits present and activating GW) Show successively the spirals corresponding to your former choices of number of iterates.
St.: (doing as told, see Figure 14 (a)) I need to have a closer look to see how the spiral approaches the intersection point. May I use the zoom facility?
Pr.: Indeed. Observe that a zooming operation means rescaling by the same amount abscissas and ordinates like seeing the intersection point as using magnifying glasses of increasing power.
St.: (see Figure 14 (b)) The more times I perform the zoom facility the closer I am to the intersection point and the closer comes the spiral to it.
Pr.: Do you see the corresponding iterate1.41421 356 in your visual field?
St.: Well, according to our former table (Figure 13), I need to consider 17 iterations (manipulating the tool, see Figure 14 (c))
Pr.: And what do the messages in the tool tell you?
St.: On one hand, the number of steps of the spiral, that is iterations, I do not see; the first ones are missing. On the other hand, the scale which has been used: ¿What means the "**e**" that appears in the scale?
Pr.: Remember that "**e**" is the exponent of ten. For example 5e-004 is the same that 5×10^{-4} that is $1/2\times10^{-3}$. Please, have a look to the third column in Figure 13.
St.: The third column marks the difference in absolute value between the n-th iteration and the actual value (which I ignore but the machine apparently knows).
Pr.: Once you have decided on the length of the spiral, that is, the number of iterations to be performed, can you guess which scale do you have to choose to see the last iterate?
St.: This numerical entry says it all: it is the scale to be chosen. That is, if n =17, then the scale has to be 10^{-7} more or less, because I can see four iterations on the screen.

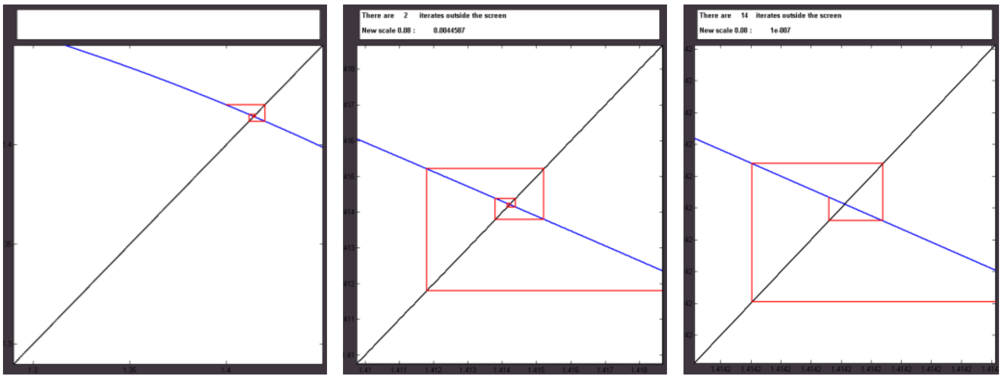


Figure 14a Figure 14b Figure 14c

Pr.: Are you confident that zooming appropriately (that is, no matter how small you choose scale) will the spiral appear in your visual field?
St.: It seems reasonable to assume that.
Pr.: In the process of zooming, does the number of those steps you cannot see change?
St.: (checking with more iterations, see Figure 15) Well, the more I zoom, the bigger is the number of those unseen.

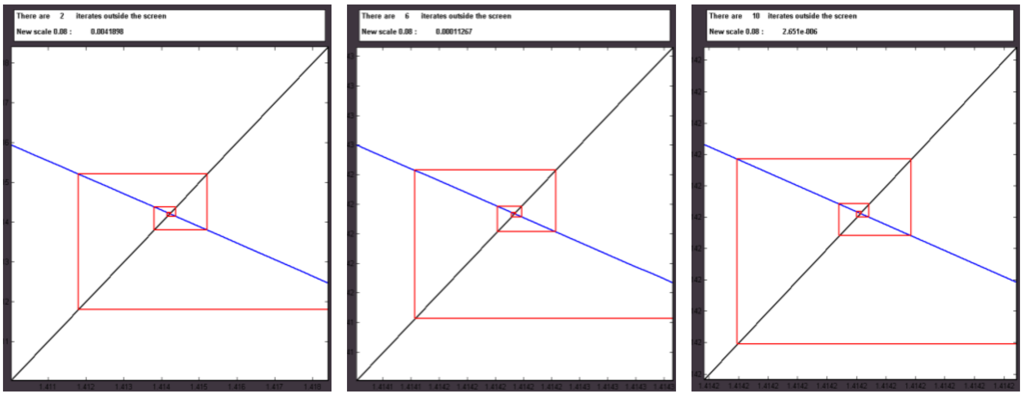


Figure 15

Pr.: Thus, there is relationship between chosen scale …

St.: A number which I am selecting smaller and smaller each time…

Pr.: And the number of iterations absent from your visual field …

St.: An integer which is growing.

Pr.: This is exactly what 'approach' or 'approximate' means in mathematics. The numbers you see in the second column of CW approach √2 in the sense we mentioned.

St.: Numbers in CW approaching √2 are explained in terms of changing scale and iterations not present in GW?

Second Step: Now we explain that what we have done is not a simple idea. As a matter of fact it took some time in the history of mathematics to arrive to a formulation of what 'approach' meant. But to prepare our student to the study of convergence, we need to eliminate expressions such as 'zoom' and 'visual field' from our considerations if we want a purely algebraic definition of what means that an orbit denoted by $x_1,x_2,x_3,\ldots$(as those shown in CW) approaches √2.

Pr.: We have three tasks: first, we need to find algebraic entities equivalent to 'zoom' or 'scale' and 'iterations not present'; secondly, the actions you have been performing on scales and, last, articulate the dependence between them.

St.: 'Zoom' can be substituted by numbers such as 0.005 and so on.

Pr.: Yes, but a single number won't do, because your actions involve changing its magnitude at will.

St.: Take a generic number, any number. Use a letter to represent it, you know, like a variable.

Pr.: A positive number?

St.: Yes, obviously.

Pr.: Use a letter like the Greek epsilon ε to denote it. First task completed. What about the actions you have to perform on it?

St.: Do you mean taking ε smaller and smaller? Something as '*every time I choose a number ε which I take as small as I desire…*'

Pr.: Anyway, this is plain English. How can you use mathematical language to describe that you have unending possibilities of choice?

St.: I see, you want me to use quantifiers.

Pr.: Yes, that is what they are for. Shall you go for the universal or for existential quantifier?

St.: (hesitating) Since I can choose epsilons at will, it has to be the universal one. Then '*for every positive number ε as small as wanted …*'

Pr.: Is '*for every positive number'* not enough? It encapsulates of possible choices.
St.: Yes, all right. Then, using the universal quantifier, *'$\forall\varepsilon>0$'*
Pr.: A notable shortcut to describe an infinity of choices. Don't you think? Second task completed. Now, what happens then '$\forall\varepsilon>0$'?
St.: *There are a finite number of iterations outside the visual zone*. Yes, yes I know, hardly a mathematical statement. Now I need the existential quantifier for an integer …
Pr.: Call it N
St.: *'$\forall\varepsilon>0$ $\exists N$ ...'*
Pr.: Is N fixed or is it variable?
St.: Once ε is chosen, N is fixed.
Pr.: But, since ε can take different values …
St.: So does N.
Pr.: That is, the value assigned to N depends on the one assigned to ε.
St.: Right. Where were we? '$\forall\varepsilon >0$ $\exists N$ …' *such that all posterior steps are in the visual field*. How do I translate those words algebraically?
Pr.: The visual field is determined by the scale chosen. What does it mean in numerical terms that, chosen a scale like let us say 0.005, the spiral continues its progress where you can see it? Try to describe this behavior in terms of the corresponding abscissas of the iterates, instead of referring to the points where the spiral changes direction.
St.: Horizontally, you mean? From position N onwards the iterates …
Pr.: Call them x_n
St.: For n larger than N, the iterates x_n are in the window, hence the difference between them and the center of the interval, that is $\sqrt{2}$, is smaller than the scale chosen. Well, I may write $x_n-\sqrt{2} < 0.005$
Pr.: Yes, but take into account that if x_n is smaller than $\sqrt{2}$, its difference is negative. Thus, writing $x_n-\sqrt{2} < 0.005$ does not ensure that x_n stays close to $\sqrt{2}$.
St.: I see, iterates approach $\sqrt{2}$ from both sides. To be sure that x_n stays that close to $\sqrt{2}$ we better use absolute values and write $| x_n-\sqrt{2} | < 0.005$
Pr.: Good. Let us forget about 0.005 and proceed as general as possible.
St.: Well, I substitute 0.005 by an ε and we get *'$\forall\varepsilon>0$ $\exists N$ $| x_n-\sqrt{2} | < \varepsilon$'*
Pr.: We are almost done. This statement holds for values of n such as?
St.: As said, for values of n larger than N
Pr.: For all of those n or only for some of them?
St.: For all of them.
Pr.: Somewhere in the statement should something involving '$n > N$' be written. Isn't it?
St.: Quantifiers again and the universal one should be: that is, $\forall n > N$. Its placement should be at the end of the statement. It should read *'$\forall\varepsilon>0$ $\exists N$ $| x_n-\sqrt{2} | < \varepsilon$ $\forall n > N$'*
Pr.: Well done. That is the algebraic-logic formulation of the sequence of iterates x_n approaching $\sqrt{2}$ and hence $\sqrt{2}$ appears as what mathematicians called the **limit** of the sequence of iterates, that is, the orbit. The formalized statement consists in three inequalities bonded by logical quantifiers.

Third Step: To check the validity of the statement *'$\forall\varepsilon>0$ $\exists N$ $| x_n-\sqrt{2} | < \varepsilon$ $\forall n > N$'*, we invite him to play our previous quantifiers game. Since the statement begins with the universal quantifier, the interviewer starts the attack. CW is activated and 14 digits are allowed.

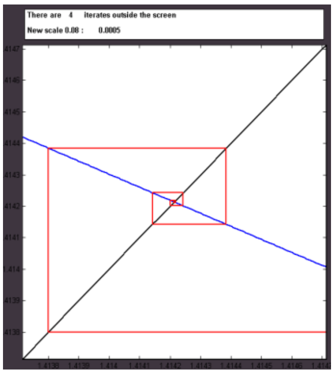


Figure 16

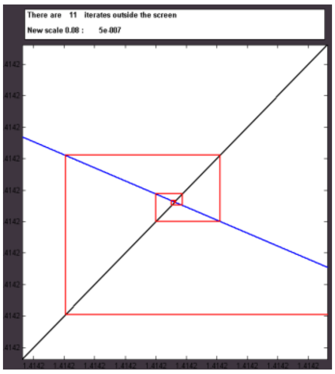


Figure 17

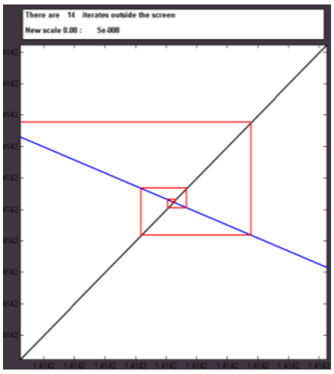


Figure 18

Pr.: I attack with $\varepsilon = \frac{1}{2} \cdot 10^{-3} = 0.0005$
St.: From GW, I defend with N = 4 (Figure 16). From the 4-th iteration, all iterations else will be inside the screen.
Pr.: The 5-th iterate is 1.4141426562317. How many of its decimal digits are in the actual value of square root of 2?
St.: If the difference in absolute value between the actual value and the 5-th iterate is less than 0.0005, I can be sure that square root of 2 reads as 1.414 and all those digits are exact.
Pr.: I attack with $\varepsilon = \frac{1}{2} \cdot 10^{-6} = 0.0000005$
St.: I defend with N = 11 (Figure 17).
Pr.: What do have in common the 12-th iterate and square root of 2?
St.: Six decimal digits are common, hence I know square root of 2 to be at least 1.414213
Pr.: I attack with $\varepsilon = \frac{1}{2} \cdot 10^{-7} = 0.00000005$
St.: I defend with N = 14, and the 15-th iterate has in common seven decimal digits with square root of 2. I know square root of 2 to be at least 1.4142135 (Figure 18).
Pr.: Are you confident that you can defend to any attack I may launch?
St.: If the spiral behaves accordingly in the long run, no doubt about it. I win, then and the statement is true.
Pr.: What is true?
St.: That the sequence of iterates generated by our procedure and shown in the second column of CW approaches $\sqrt{2}$ or, seen as a spiral in GW, encroach the point ($\sqrt{2}$, $\sqrt{2}$).
Pr.: More succinct, $\sqrt{2}$ is the limit of the orbit.

Now we revisit our previous procedures *2x-2/x* to *2/x* and we play (a very short) game with respect to the defense of the statement '$\forall \varepsilon > 0\ \exists N \mid x_n - \sqrt{2} \mid < \varepsilon\ \forall n > N$' to check that it is not valid.

Pr: (see Figure 19) I attack with $\varepsilon = 0.05$
St.: I cannot find any N to defend the statement because the stairs go the wrong way. When I enlarge the image, only the first iterations are inside the screen. The difference between the shown iterations and $\sqrt{2}$ gets bigger and bigger.
Pr.: And… the conclusion?
St.: The statement is false in this case.

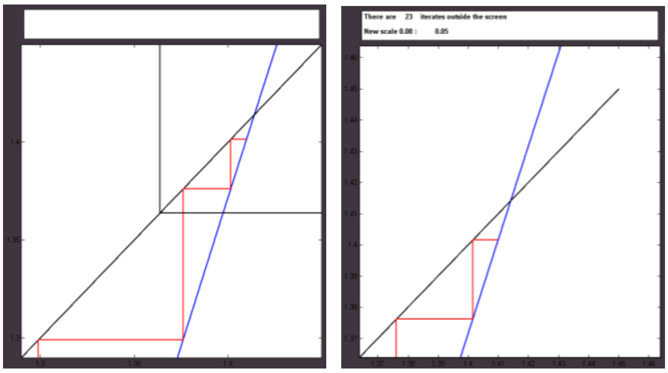


Figure 19

Pr.: Now we go to see *2/x*? I attack with $\varepsilon = 0.05$.
St.: I defend with N = 0, all iterates are inside the screen. But I see trouble coming with further attacks (Figure 20).
Pr.: I attack with $\varepsilon = 0.0005$.
St.: The ε is smaller than the half of the side of the red square and all iterates are outside the screen, I cannot defend the statement in this case either (Figure 21).

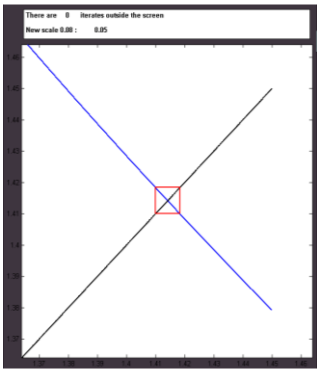


Figure 20

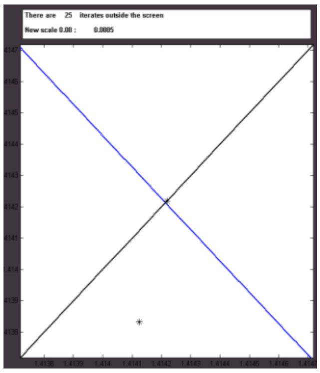


Figure 21

We stress the importance of the order of quantifiers

Pr.: Does our statement differ from the following one ‘$\exists N\ \forall\varepsilon>0\ |\ x_n-\sqrt{2}\ | < \varepsilon\ \forall\ n > N$’?
St.: Well, we know that altering the order of the quantifiers, alters also the meaning of the statement. In our case, it is stated that there is an integer N such that something happens, isn`t it?
Pr.: Yes. N stands for a position in the sequence of iterates and first you have to analyze what ‘something’ means.
St.: Something means: iterates (from the N-th onwards) are all in the visual zone, no matter how small is the scale used, which makes no sense.
Pr.: Unless …
St.: All iterates are the same number and coincide with $\sqrt{2}$.

Recapitulating: armed with an ‘working’ iterative procedure and a suitable starting approximation x_1 we can generate an orbit $x_1, x_2, x_3, \ldots$. We want him to be sure that in order to answer the question **‘What is √2?’** he/she needs to use a recently constructed definition by saying that $\sqrt{2}$ is the **limit** of the orbit, that is, the only number satisfying statement ‘$\forall\varepsilon>0\ \exists N\ |\ x_n-\sqrt{2}\ | < \varepsilon$

$\forall n > N$' whose digits can be gradually calculated by implementing the procedure. Collaterally, we want him/her to state explicitly that there are several different procedures available to perform this task providing different orbits, some better than others in the sense of providing exact digits quicker and that not every possible admissible procedure (in the sense of originating from equation $x^2 = 2$) provides an orbit verifying the statement and that selection of a suitable procedure has to be made in accordance to its local inclination, which easily translates itself in terms of derivatives.

Report. From the remaining nine of our initial sample, three were unable to conquer the first step of this Phase by their inability to understand the dynamics of the concept of convergence (as illustrated by a series of images provided by the tool) which translates in logically conditioned statements about closeness and the behavior of the spiral. Those six students who understood what was going on sailed, not without difficulty, through the remaining two steps and had no trouble dealing with the game.

6. Coda

We have framed our research in van Hiele's model, van Hiele (1986), originally applied to the study of geometrical properties, because, on one hand, the model is more concerned on how students think about a specific topic than with the topic itself and, on the other, because the model mimics the genesis of some mathematical concepts: first, the discovery of isolated phenomena; second, the acknowledgement of certain characteristics common to all of them; third, the search for new objects, their study and classification and, fourth, through consideration of examples and counterexamples to proposed definitions, the emergence of definitive formulations. The model provides a description of the learning process, by postulating the existence of levels of reasoning (not identified with computational skills) classified as Level 1 (Visual Recognition), Level 2 (Analysis), Level 3 (Classification and Relation) and Level 4 (Formal Deduction). In Level 1 students are guided by a series of visual characteristics and lead by their intuition. In Level 2, individuals notice the existence of a network of relationships. This is the first level of reasoning that can be called "mathematical" because students are able to describe and generalize through observation and manipulation properties that they still do not know. Reasoning in Level 3 is related to the structure of the second level and conclusions are no longer based on the existence or non-existence of links in the network of relationships of the second level, but rather on existing connections between those links. Level 4 speaks for itself. To be considered within van Hiele's model: (i) levels must be hierarchical, recursive, and sequential (ii) levels must be formulated so that they include a progression in the level of reasoning as a result of a gradual process, resulting from learning experiences (iii) tests designed for the detection of levels should take into account the existing relationships among levels and the language used by apprentices and (iv) the fundamental objective of the design must be the detection of levels of reasoning, without confusing them with levels of computational skill or previous knowledge.

The learning process was structured as a Socratic semi-structured interview and its design had to comply with (a) its purpose as stated at the beginning of this article and (b) the constraints posed by the model. Concerning (a), what is aimed at is the slow construction of a concept image of convergence, which needs the amalgamation of disparate ideas (some already known by the student, some vague intuitions) such as the arithmetic nature of an irrational number, the loss of exactness and how to recuperate it: the notion of approximating sequence of rational numbers, how is it generated by iteration, the notion of function as a process (to deal with iteration) and how attach meaning to the term 'approximate'. Concerning (b), its goal is the detection of the levels postulated by the model which can be considered as the boundaries of several cycles of discovery-acceptance-confidence-doubt-reconstruction: language used by interviewees plays a role (see (iii) above), hence the need of profuse talking and transcription analysis and, in order to comply with (iv) above, calculations are left to a machine.

Our inquiries would not have gone too far without a computer-generated tool based on a powerful mathematical assistant designed as a Graphic User Interface (GUI) allowing the combination of multiple windows (graphic and computational) with interactive capabilities to show different representations of the concept and simulate dynamism. As a general rule, it is clear to us that its use may complicate learning from a didactic point of view, since it is mainly a materialization of symbolic technology and hence changes the material to be taught by transposing everything to a computational problem, but its careful use can provide a richly textured view of the problem in question. On one hand, left alone, adverse effects in creativity and problem solving are bound to arise, putting the student in extreme dependence to the machine, anesthetizing his impulses to look for other representations of the problem. On the other hand, as said, we see only what we understand and to address potentially damaging effects, the use of the GUI needs the company and the complicity of a well-structured dialog to ensure success. The tool plays then a dual role: as amplifier of computational capabilities and, through the generation of dynamical representations, as simulator of thought processes. The tool, apart from easing the burden of calculations, promotes higher levels of engagement encouraging students' participation, provides them with numerical data and visual realizations which, paradoxically as it may sound, facilitate abstraction by linking the processes of discovery, understanding and conceptualization.

Although we reported periodically on failure along the experience and its causes, we leave for a work in progress to explain how it all fits within this educative model, why certain students failed to reach the higher levels of the model, how we extended our study to a larger sample of students via a multiple choice test designed as a spin-off of the interview and why such a limited degree of success was not unreasonable. It was not our purpose to leave the job of analysis of data to the reader, but extension considerations lead us to present only our empirical work as we feel careful observation has value as an end in itself in helping students to see their misconceptions and helping us to remember that abstractions of mathematics are not trivial and do require time and effort to acquire. The fact that we turned the interview into a learning process revealed some general patterns in the thinking and misconceptions of our students that lead us to a better understanding of how they learn limiting processes. For those readers more interested in the methodological aspects of our strategy than in its theoretical foundation, this article may stand on its own by providing hints on how to proceed individually with students or even on how to adapt its contents as classroom material by showing what cognitive obstacles are expected to appear and how to deal with or circumvent them.

7. References

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